1. Prove or Disprove.
(a) A group of order 35 is cyclic.
Solution: Yes, group of order 35 is cyclic. See page no. 201, "Comtemporary Abstract Algebra" by Joseph A. Gallian.

(b) There exists exactly 90 elements of order 7 in a simple group of order 105.
Solution: Since $105 = 3 \times 5 \times 7$, by Sylow's theorem the number of 7-Sylow subgroups is either 1 or 15. If there are 15 such subgroups then we get $15 \times 6 = 90$ elements of order 7. The number of 5-Sylow subgroups is either 1 or 21. If there are 21 5-sylow subgroups, then we $21.4 = 84$ number of elements of order 5. This is impossible in a group of order 105. Thus there is a unique subgroup G_5 of order 5, and it is normal.

(c) If $H \trianglelefteq G$, such that H intersects the commutator subgroup of G trivially, then $H \subseteq Z(G)$.
Solution: Suppose $G' = \{a^{-1}b^{-1}ab : a,\ b \in G\}$ be the commutator subgroup of G. Let $h \in H$. Then for any $x \in G$, we have $h^{-1}x^{-1}hx \in H \cap G' = \{e\}$. Which implies $hx = xh$. Therefore $h \in Z(G)$.

(d) The automorphism group of $Z2 \times Z2$ is $Z6$.
Solution: No. The automorphism group of $Z2 \times Z2$ is isomorphic to S_3, which is non-abelian.

(e) S_4 and D_{24} are isomorphic.
Solution: No. We have $D_24 =< x,\ y|x^{12} = 1 = y^2,\ xy = x^{11}y >$ and the order of x is 12. The orders of the elements of S_4 depend on their cycle type only:

$4 = 4$ yields order 4
$4 = 1 + 3$ yields order 3
$4 = 2 + 2$ yields order 2
$4 = 2 + 1 + 1$ yields order 2
$4 = 1 + 1 + 1 + 1$ yields order 1. So S_4 has no element of order 12. Therefore S_4, D_{24} are not isomorphic.

2. (a) Show that the commutator subgroup of a group G is a normal subgroup of G.
Solution: Let $G' = \{aba^{-1}b^{-1} : a,\ b \in G\}$ be the commutator subgroup of G. For any $x \in G$, $x(aba^{-1}b^{-1})x^{-1} = (xax^{-1})(xbx^{-1})(xa^{-1}x^{-1})(xb^{-1}x^{-1}) = (xax^{-1})(xbx^{-1})(xax^{-1})^{-1}(xbx^{-1})^{-1} \in G'$.

(b) Show that the commutator subgroup of S_n is A_n, for all $n \geq 3$.
Solution: Since S_n/A_n is commutative, the commutator subgroup S'_n is contained in A_n. Conversely, we have $(12)(13)(12)^{-1}(13)^{-1} = (123)$, showing that every 3-cycle is in S'_n. Since A_n is generated by 3-cycles, $S'_n = A_n$.

3. (a) Let G be a group acting on a set X and let $x \in X$. Let $\text{Orb}(x)$ denote the orbit of x and G_x denote the stabiliser of x. Show that $|Orb(x)| = |G : G_x|$.
Solution: Orbit-Stabilizer theorem. See page no. 139, "Comtemporary Abstract Algebra" by Joseph A. Gallian.

(b) Let p be a prime, $o(G) = p^n$ for some $n \geq 1$ and let G act on a finite set X.
Let $X_0 = \{x \in X \text{such that} g.x = x \forall g \in G\}$ be the fixed point set. Show that $|X| \equiv |X_0| mod p$.
Solution: See page no. 492, "Comtemporary Abstract Algebra" by Joseph A. Gallian.

4. (a) State Sylow's theorems.
Solution: See Theorem 24.3, 24.4, 24.5. (page 399, 400, 401) in "Comtemporary Abstract Algebra" by Joseph A. Gallian.
(b) Let $o(G) = pqr$ where $p; q; r$ are primes with $p < q < r$. Show that G has a normal Sylow subgroup for either$p; q$ or r.
Solution: Let n_s be the number of s-Sylow subgroups of G, where $s = p,\ q,\ r$. For each s, by the Sylow Theorems n_s divides $|G|$ and $n_s = 1 + ks$ for some integer k. In particular s does not divide n_s. Suppose that no s-Sylow subgroup is normal. Then $n_s \geq 1 + s$ for $s = p,\ q,\ r$. Since n_p is among $q,\ r,\ qr$ and $q < r$, so $n_p \geq q$. Since n_q is among p, r, pr and $p < q,\ r \leq qr$, so $n_q \geq r$. Since n_r is among $p; q; pq$ and $p, q < r$ we have $n_r = pq$. Since each s-Sylow subgroup of G is cyclic of prime order, each of these subgroups has $s - 1$ elements of order s. Counting the elements of order p, q and r respectively gives

$$q(p-1) + r(q-1) + pq(r-1) \leq pqr,$$

so that $qr \leq q + r$. This implies $qr \leq q + r \leq 2r$, so $qr \leq 2r$. i.e $q \leq 2$ which is a contradiction.

5. Determine the Sylow subgroups of A_5.
Solution: $|A_5| = 60 = 2^2 \times 3 \times 5$. A_5 has five 2-sylow subgroups of order 4, ten 3-sylow subgroups of order 3, six 5-sylow subgroups of order 5.

6. Let G be a finite abelian group. Show that G is the (internal) direct product of its Sylow subgroups.
Solution: See page no. 109, "Topics in Algebra" by i.n. herstein.
7. (a) Define external semidirect product of two groups H and K.
Solution: H is a group and K is a group acting on H; in other words, there is a group homomorphism $\rho : K \to \mathrm{Aut}(H)$, from K to the automorphism group of H. The external semidirect product G of H and K, denoted $H \rtimes K$ is, as a set, the Cartesian product $H \times K$, with multiplication given by the rule:

$(a, b)(a', b') = (a(\rho(b)(a')), bb')$.

Writing the action $\rho(b)a' = b \cdot a'$, we get $(a, b)(a', b') = (a(b \cdot a'), bb')$

(b) Classify all groups of order 12 where the Sylow 3-subgroup is normal.
Solution: Upto isomorphism, there are three types groups of order 12 where the Sylow 3-subgroup is normal. (1) cyclic groups, Z_{12}. (2) Abelian groups, $Z_6 \bigotimes Z_2$. (3) Non-abelian groups, Dihedral group D_{12}, dicyclic group Dic_{12}.